## Volumes of revolution 5A

**1 a** $V = \pi\int_0^2 \left(10x^2\right)^2 \text{ d}x$
$= \pi\int_0^2 100x^4 \text{ d}x$
$= \pi\left[20x^5\right]_0^2$
$= \pi(640 - 0)$
$= 640\pi$

**b** $V = \pi\int_3^5 (5-x)^2 \text{ d}x$
$= \pi\int_3^5 \left(25 - 10x + x^2\right) \text{ d}x$
$= \pi\left[25x - 5x^2 + \frac{x^3}{3}\right]_3^5$
$= \pi\left(\left(125 - 125 + \frac{125}{3}\right) - (75 - 45 + 9)\right)$
$= \pi\left(\frac{125}{3} - 39\right)$
$= \frac{8}{3}\pi$

**c** $V = \pi\int_2^{10} \left(\sqrt{x}\right)^2 \text{ d}x$
$= \pi\int_2^{10} x \text{ d}x$
$= \pi\left[\frac{x^2}{2}\right]_2^{10}$
$= \pi\left(\frac{100}{2} - \frac{4}{2}\right)$
$= 48\pi$

**d** $V = \pi\int_1^2 \left(1 + \frac{1}{x^2}\right)^2 \text{ d}x$
$V = \pi\int_1^2 \left(1 + \frac{2}{x^2} + \frac{1}{x^4}\right) \text{d}x$
$= \pi\left[x - \frac{2}{x} - \frac{1}{3x^3}\right]_1^2$
$= \pi\left(\left(2 - 1 - \frac{1}{24}\right) - \left(1 - 2 - \frac{1}{3}\right)\right)$
$= \pi\left(\frac{23}{24} + \frac{4}{3}\right)$
$= \frac{55}{24}\pi$

**2** $5 + 4x - x^2 = 0$
$(5-x)(1+x) = 0$
$x > 0 \Rightarrow x = 5$
$V = \pi\int_0^5 \left(5 + 4x - x^2\right)^2 \text{ d}x$
$= \pi\int_0^5 \left(25 + 40x + 6x^2 - 8x^3 + x^4\right) \text{d}x$
$= \pi\left[25x + 20x^2 + 2x^3 - 2x^4 + \frac{x^5}{5}\right]_0^5$
$= \pi(125 + 500 + 250 - 1250 + 625)$
$= 250\pi$

**3** $V = \pi\int_1^8 \left(3 - \sqrt[3]{x}\right)^2 \text{ d}x$
$V = \pi\int_1^8 \left(9 - 6x^{\frac{1}{3}} + x^{\frac{2}{3}}\right) \text{d}x$
$= \pi\left[9x - \frac{9}{2}x^{\frac{4}{3}} + \frac{3}{5}x^{\frac{5}{3}}\right]_1^8$
$= \pi\left(\left(72 - 72 + \frac{96}{5}\right) - \left(9 - \frac{9}{2} + \frac{3}{5}\right)\right)$
$= \pi\left(\frac{96}{5} - \frac{51}{10}\right)$
$= \frac{141}{10}\pi$

**4** $\sqrt{x+2} = 0 \Rightarrow x = -2$
$V = \pi\int_{-2}^2 \left(\sqrt{x+2}\right)^2 \text{ d}x$
$= \pi\int_{-2}^2 (x+2) \text{ d}x$
$= \pi\left[\frac{x^2}{2} + 2x\right]_{-2}^2$
$= \pi((2+4) - (2-4))$
$= 8\pi$

**5 a** $9x^{\frac{3}{2}} - 3x^{\frac{5}{2}} = 0$
$3x^{\frac{3}{2}}(3-x) = 0$
$x = 0$ or $x = 3$
Coordinates of $A$ are (3, 0)

5 **b** $V = \pi\int_0^3 \left(9x^{\frac{3}{2}} - 3x^{\frac{5}{2}}\right)^2 \, dx$

$= \pi\int_0^3 \left(81x^3 - 54x^4 + 9x^5\right) dx$

$= \pi\left[\frac{81}{4}x^4 - \frac{54}{5}x^5 + \frac{3}{2}x^6\right]_0^3$

$= \pi\left(\frac{6561}{4} - \frac{13122}{5} + \frac{2187}{2}\right)$

$= \frac{2187}{20}\pi$

**6** $\frac{\sqrt{3x^4 - 3}}{x^3} = 0 \Rightarrow x = \pm 1$

From the graph $x > 0$ so $C$ cuts the $x$-axis at $x = 1$.

$V = \pi\int_1^6 \left(\frac{\sqrt{3x^4 - 3}}{x^3}\right)^2 dx$

$= \pi\int_1^6 \left(\frac{3x^4 - 3}{x^6}\right) dx$

$= \pi\int_1^6 \left(\frac{3}{x^2} - \frac{3}{x^6}\right) dx$

$= \pi\left[-\frac{3}{x} + \frac{3}{5x^5}\right]_1^6$

$= \pi\left(\left(-\frac{3}{6} + \frac{3}{38\,880}\right) - \left(-3 + \frac{3}{5}\right)\right)$

$= 5.97$ (3 s.f.)

**7** $5y^2 - x^3 = 2x - 3$

$y^2 = \frac{1}{5}\left(x^3 + 2x - 3\right)$

$V = \frac{\pi}{5}\int_1^4 \left(x^3 + 2x - 3\right) dx$

$= \frac{\pi}{5}\left[\frac{x^4}{4} + x^2 - 3x\right]_1^4$

$= \frac{\pi}{5}\left((64 + 16 - 12) - \left(\frac{1}{4} + 1 - 3\right)\right)$

$= \frac{\pi}{5}\left(68 + \frac{7}{4}\right)$

$= \frac{279}{20}\pi$

**8** $V = \pi\int_a^2 \left(x\sqrt{4 - x^2}\right)^2 dx$

$= \pi\int_a^2 x^2\left(4 - x^2\right) dx$

$= \pi\int_a^2 \left(4x^2 - x^4\right) dx$

$= \pi\left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_a^2$

$= \pi\left(\left(\frac{32}{3} - \frac{32}{5}\right) - \left(\frac{4a^3}{3} - \frac{a^5}{5}\right)\right)$

$= \pi\left(\frac{64}{15} - \frac{20a^3 - 3a^5}{15}\right)$

$= \frac{\pi}{15}\left(64 - 20a^3 + 3a^5\right)$

But $V = \frac{657}{160}\pi$ so $64 - 20a^3 + 3a^5 = \frac{15 \times 657}{160}$

$3a^5 - 20a^3 + \frac{77}{32} = 0$

For $0 < a < 2$,

try $a = 1 \Rightarrow -\frac{467}{32} \neq 0$,

try $a = \frac{1}{2} \Rightarrow 0$

Hence, $\left(a - \frac{1}{2}\right)$ is a factor of $3a^5 - 20a^3 + \frac{77}{32}$.

Hence, a solution of $3a^5 - 20a^3 + \frac{77}{32} = 0$ is $a = \frac{1}{2}$.

**9** Equation of line is $y = r$

$V = \pi\int_0^h r^2 \, dx$

$= \pi\left[r^2 x\right]_0^h$

$= \pi r^2 h$

**Challenge**

$x^2-7x+10=0$

$(x-2)(x-5)=0 \Rightarrow x=2$ or $x=5$

So the $x$ coordinates of the points where the curve touches the $x$-axis are 2 and 5.

Splitting the region into 3 sections, $R_1$, $R_2$ and $R_3$

$R_1$ for $1 \leqslant x \leqslant 2$

$R_2$ for $2 < x \leqslant 5$

$R_3$ for $5 < x \leqslant 6$

Volume $V_1$ is generated by $R_1$, etc.

$$\begin{aligned}
V_1 &= \pi\int_1^2 \left(x^2-7x+10\right)^2\ dx \\
&= \pi\int_1^2 \left(x^4-14x^3+69x^2-140x+100\right) dx \\
&= \pi\left[\frac{x^5}{5}-\frac{7x^4}{2}+23x^3-70x^2+100x\right]_1^2 \\
&= \pi\left(\frac{272}{5}-\frac{497}{10}\right)=\frac{47}{10}\pi
\end{aligned}$$

$$\begin{aligned}
V_2 &= \pi\int_2^5 \left(x^2-7x+10\right)^2\ dx \\
&= \pi\int_2^5 \left(x^4-14x^3+69x^2-140x+100\right) dx \\
&= \pi\left[\frac{x^5}{5}-\frac{7x^4}{2}+23x^3-70x^2+100x\right]_2^5 \\
&= \pi\left(\frac{125}{2}-\frac{272}{5}\right)=\frac{81}{10}\pi
\end{aligned}$$

$V_3=V_1$ using the symmetry of the curve.

So total volume generated $=\pi\left(\dfrac{47}{10}+\dfrac{81}{10}+\dfrac{47}{10}\right)$

$$=\frac{175}{10}\pi$$

$$=\frac{35}{2}\pi$$